Iterated integrals of Faulhaber polynomials and some properties of their roots

Piotr LORENC, Jakub Jan LUDEW,
Mariusz PLESZCZYŃSKI, Alicja SAMULEWICZ
and Roman WITUŁA

Abstract. This paper is devoted to the decomposition of Faulhaber polynomials $S_m(x)$ into iterated integrals. It has been noticed, by the way, that these polynomials provide the primary example of so-called semihyperbolic polynomials. There are also presented some interesting properties concerning the number of complex roots (as well as the real roots) of translated Faulhaber polynomials $S_m^*(x) = S_m(x) + \frac{B_{m+1}}{m+1}$, where B_{m+1} denotes the respective Bernoulli number for each $m \leq 1024$. Three of these properties are particularly intriguing. 1° If a polynomial $S_{m+1}^*(x)$ has more nonreal complex roots than $S_m^*(x)$ does then their number is always greater than 4 for all $m \leq 1023$. 2° If S_n^*, $S_{n+1}^*, \ldots, S_{n+k}^*$ have the same number of nonreal complex roots but S_{n+k+1}^* possesses more such roots and S_{n-1}^* possesses less such roots then either $k = 4$ or $k = 5$ for all $k, n \in \mathbb{N}$, $n + k \leq 1024$. 3° If S_n^* has 12 nonreal complex roots less than S_{n+k}^* and simultaneously 16 less than S_{n+k+1}^* and 4 more than S_{n-1}^*, then either $k = 19$ or $k = 20$ for every $k, n \in \mathbb{N}$, $11 \leq n$ and $k + n \leq 1024$. The authors are convinced – and the results of numerical calculations seem to confirm this opinion – that all the properties hold for infinitely many values of $n \in \mathbb{N}$. Similar observations concerning the real roots of polynomials S_n^* are also presented in the paper. Furthermore, the envelopes of the complex roots distributions of polynomials $S_m^*(x)$ are generated for some special values of m.

Keywords: Bernoulli numbers, Faulhaber polynomials, translated Faulhaber polynomials, iterated integrals.

2010 Mathematics Subject Classification: 11B68, 12D05, 12D10, 26C05, 30C15.

J.J. Ludew, M. Pleszczyński, A. Samulewicz, R. Wituła
Institute of Mathematics, Silesian University of Technology, Kaszubska 23, 44-100 Gliwice, Poland,
e-mail: {jakub.ludew, mariusz.pleszczynski, alicja.samulewicz, roman.witula}@polsl.pl

While creating this paper P. Lorenc was a student of the master's degree study.

R. Wituła, B. Bajorska-Harapińska, E. Hetmaniok, D. Słota, T. Trawiński (eds.), *Selected Problems on Experimental Mathematics*. Wydawnictwo Politechniki Śląskiej, Gliwice 2017, pp. 201–217.

1. Introduction

While working on the monograph [15] Wituła has formulated the following problem:

Question. Given a number $m \in \mathbb{N}$ find all sequences $\{a_k\}_{k=1}^{m+1} \subset \mathbb{R}$ such that

$$m! \int_{a_{m+1}}^{n} \mathrm{d}x_m \int_{a_m}^{x_m} \mathrm{d}x_{m-1} \ldots \int_{a_1}^{x_1} \mathrm{d}x_0 = \sum_{k=1}^{n} k^m \tag{1}$$

for every $n \in \mathbb{N}$.

The goal of this paper is to solve the above problem, or more precisely, to find its connection with the problem of the decomposition of Faulhaber polynomials $S_m(x)$ into iterated integrals. To this aim recall that polynomials $S_m(x)$ are defined in the following way

$$S_m(x) = \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k x^{m+1-k}, \ m \in \mathbb{N}, \tag{2}$$

where B_k's are the modified Bernoulli numbers [7, 15, 16] with initial values $B_0 = 1$, $B_1 = \frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$ and $B_{2k+1} = 0$, $k \in \mathbb{N}$. These polynomials satisfy the condition

$$S_m(n) = \sum_{k=1}^{n} k^m$$

for $m, n \in \mathbb{N}$.

The Faulhaber polynomials are the primary examples of so-called semihyperbolic polynomials[1], i.e. polynomials $p \in \mathbb{R}[x]$ such that all derivatives $\frac{\mathrm{d}^k p}{\mathrm{d}x^k}$, $0 \leq k < \delta$ possess at least one real root, where δ denotes the degree of p. Note that in this case (and only in this case) there exist $\alpha_1, \alpha_2, \ldots, \alpha_\delta \in \mathbb{R}$ such that

$$A \int_{\alpha_\delta}^{x} \mathrm{d}x_\delta \int_{\alpha_{-1+\delta}}^{x_\delta} \mathrm{d}x_{-1+\delta} \cdots \int_{\alpha_1}^{x_2} \mathrm{d}x_1 = \frac{1}{\delta!}\, p(x),$$

where A denotes the leading coefficient of p (see [8][2]).

Iterated integrals of this type represent the volumes of some polyhedrons all faces of which are triangular and one coordinate of one of the vertices is a variable. However this is not the subject of consideration here[3].

---

[1] Recall that a polynomial is said to be hyperbolic provided that all its roots are real. The Faulhaber polynomials represent an example of a family of polynomials which are semihyperbolic, but not hyperbolic.

[2] Actually [8] constitutes the second part of this article.

[3] It is worth mentioning that expressing a special function as an iterated integral is studied by numerous scientists, such is in the case with the classical multiple zeta values. See also the Vinogradov-Korobov method of estimation of some exponential integrals in Ivič's monograph [9, Chapter 6].

1.1. Technical preliminaries

Consider the function

$$f_m(x) = m! \int_{a_{m+1}}^{x} \mathrm{d}x_m \int_{a_m}^{x_m} \mathrm{d}x_{m-1} \dots \int_{a_1}^{x_1} \mathrm{d}x_0,\ x \in \mathbb{R},$$

where $m \in \mathbb{N}$ and $a_1, \dots, a_{m+1}$ are reals. It is easy to see that $f_m(x)$ is a polynomial of degree $m+1$. Our main aim is to find all possible numbers $a_1, a_2, \dots a_{m+1}$ such that $f_m(x) \equiv S_m(x)$. We have

$$\frac{\mathrm{d}^p f_m(x)}{\mathrm{d}x^p} = m! \int_{a_{m+1-p}}^{x} \mathrm{d}x_{m-p} \int_{a_{m-p}}^{x_{m-p}} \mathrm{d}x_{m-1-p} \dots \int_{a_1}^{x_1} \mathrm{d}x_0$$

for every $p = 0, 1, \dots, m$, and $\dfrac{\mathrm{d}^{m+1} f_m(x)}{\mathrm{d}x^{m+1}} = m!$. Then the coefficient of x^p in $f_m(x)$ is equal to

$$\frac{1}{p!}\frac{\mathrm{d}^p f_m(x)}{\mathrm{d}x^p}\Big|_{x=0} = \frac{m!}{p!} \int_{a_{m+1-p}}^{0} \mathrm{d}x_{m-p} \int_{a_{m-p}}^{x_{m-p}} \mathrm{d}x_{m-1-p} \dots \int_{a_1}^{x_1} \mathrm{d}x_0$$

for every $p = 0, 1, \dots, m$, and $\dfrac{1}{(m+1)!}\dfrac{\mathrm{d}^{m+1} f_m(x)}{\mathrm{d}x^{m+1}}\Big|_{x=0} = \dfrac{m!}{(m+1)!} = \dfrac{1}{m+1}$. Comparing the above equalities with the corresponding coefficients of the polynomial $S_m(x)$ we get

$$m! \int_{a_{m+1}}^{0} \mathrm{d}x_m \int_{a_{m-1}}^{x_{m-1}} \mathrm{d}x_{m-2} \dots \int_{a_1}^{x_1} \mathrm{d}x_0 = 0 \tag{3}$$

and

$$\frac{m!}{p!} \int_{a_{m+1-p}}^{0} \mathrm{d}x_{m-p} \int_{a_{m-p}}^{x_{m-p}} \mathrm{d}x_{m-1-p} \dots \int_{a_1}^{x_1} \mathrm{d}x_0 = \frac{1}{m+1} \binom{m+1}{m+1-p} B_{m+1-p},$$

i.e.

$$\int_{a_{m+1-p}}^{0} \mathrm{d}x_{m-p} \int_{a_{m-p}}^{x_{m-p}} \mathrm{d}x_{m-1-p} \dots \int_{a_1}^{x_1} \mathrm{d}x_0 = \frac{B_{m+1-p}}{(m+1-p)!}$$

for every $p = 1, 2, \dots, m$. After rescaling the subscript $m+1-p := k$ we get

$$\int_{a_k}^{0} \mathrm{d}x_{k-1} \int_{a_{k-1}}^{x_{k-1}} \mathrm{d}x_{k-2} \dots \int_{a_1}^{x_1} \mathrm{d}x_0 = \frac{B_k}{k!} \tag{4}$$

for every $k = 1, 2, \dots, m$. Hence we can see that values of sought numbers a_i, $i = 1, 2, \dots, k$ do not depend on m, whenever $m \geq k$ and by some straightforward calculations we easily find $a_1 = -\dfrac{1}{2}, a_2 \in \Big\{\dfrac{1}{6}(-3 \pm \sqrt{3})\Big\}$, $a_3 \in \Big\{-1, -\dfrac{1}{2}, 0\Big\}$.

Assume that we have already found the numbers $a_1, a_2, \ldots, a_j$ for some $j \in \{1, 2, \ldots, m-1\}$. Then, if we want to find the values of a_{j+1} we must solve the equation

$$\int_y^0 \mathrm{d}x_j \int_{a_j}^{x_j} \mathrm{d}x_{j-1} \ldots \int_{a_1}^{x_1} \mathrm{d}x_0 = \frac{B_{j+1}}{(j+1)!} \tag{5}$$

for y. It is an algebraic equation of degree $j+1$. Let A_{j+1} be the set of its solutions (we assume A_{j+1} to be the set of all complex solutions since we know nothing about the real ones). To solve (5) we need the following auxiliary result.

Lemma 1.1. *For every $j = 1, 2, \ldots$ the following identity holds*

$$\int_{a_j}^{x} dx_{j-1} \ldots \int_{a_1}^{x_1} dx_0 = \frac{S_{j-1}(x)}{(j-1)!} + \frac{B_j}{j!}. \tag{6}$$

Proof. The proof follows by induction with respect to j.
We have

$$\int_{a_1}^{x} \mathrm{d}x_0 = x + \frac{B_1}{1!} = S_0(x) + \frac{B_1}{1!}.$$

Assume that for some $l \in \mathbb{N}$ we have

$$\int_{a_l}^{x} \mathrm{d}x_{l-1} \int_{a_{l-1}}^{x_{l-1}} \mathrm{d}x_{l-2} \ldots \int_{a_1}^{x_1} \mathrm{d}x_0 = \frac{S_{l-1}(x)}{(l-1)!} + \frac{B_l}{l!}.$$

Then we obtain

$$\begin{aligned}
&\int_{a_{l+1}}^{x} \mathrm{d}x_l \int_{a_l}^{x_l} \mathrm{d}x_{l-1} \ldots \int_{a_1}^{x_1} \mathrm{d}x_0 \\
&\quad = \int_{a_{l+1}}^{0} \mathrm{d}x_l \int_{a_l}^{x_l} \mathrm{d}x_{l-1} \ldots \int_{a_1}^{x_1} \mathrm{d}x_0 + \int_0^x \mathrm{d}x_l \int_{a_l}^{x_l} \mathrm{d}x_{l-1} \ldots \int_{a_1}^{x_1} \mathrm{d}x_0 \\
&\quad \overset{(6)}{=} \frac{B_{l+1}}{(l+1)!} + \int_0^x \Big(\frac{S_{l-1}(x_l)}{(l-1)!} + \frac{B_l}{l!} \Big) \mathrm{d}x_l.
\end{aligned} \tag{7}$$

By (2) we have

$$\begin{aligned}
\int \frac{S_{l-1}(t)}{(l-1)!} \mathrm{d}t &= \frac{1}{(l-1)!} \int \frac{1}{l} \sum_{k=0}^{l-1} \binom{l}{k} B_k t^{l-k} \mathrm{d}t = \frac{1}{l!} \sum_{k=0}^{l-1} \binom{l}{k} B_k \frac{t^{l+1-k}}{l+1-k} + C \\
&= \frac{1}{l!} \left(\left(\frac{1}{l+1} \sum_{k=0}^{l} \binom{l+1}{k} B_k t^{l+1-k} \right) - B_l t \right) + C = \frac{S_l(t) - B_l t}{l!} + C,
\end{aligned} \tag{8}$$

and hence

$$\int_0^x \Big(\frac{S_{l-1}(x_l)}{(l-1)!} + \frac{B_l}{l!} \Big) \mathrm{d}x_l = \frac{S_l(x) - S_l(0)}{l!} = \frac{S_l(x)}{l!}. \tag{9}$$

Therefore equality (6) is satisfied for $j = l+1$. This completes the proof. □

Corollary 1.2. *For every $j = 1, 2, \ldots$ the equation (5) is equivalent to*

$$S_j(y) = -\frac{B_{j+1}}{j+1}.$$

Proof. From (6) and (9) we get

$$\int_y^0 \mathrm{d}x_j \int_{a_j}^{x_j} \mathrm{d}x_{j-1} \ldots \int_{a_1}^{x_1} \mathrm{d}x_0 \\ = \int_y^0 \left(\frac{S_{j-1}(x_j)}{(j-1)!} + \frac{B_j}{j!} \right) \mathrm{d}x_j = \frac{S_j(0) - S_j(y)}{j!} = -\frac{S_j(y)}{j!}. \tag{10}$$

□

The only remaining problem is to determine the value of a_{m+1}. Note that $f_m(a_{m+1}) = 0$, so a_{m+1} must satisfy the condition

$$S_m(a_{m+1}) = 0. \tag{11}$$

Thus a_{m+1} could be any real root of polynomial $S_m(x)$. Since for any $m \in \mathbb{N}$ we have $S_m(0) = S_m(-1) = 0$, therefore 0 or -1 may be possible values of a_{m+1}.

2. The form of sets A_j

Fix $m \in \mathbb{N}$ and set $A_1 = \{\frac{-1}{2}\}$. For every $j = 1, 2, \ldots, m-1$ determine the set A_{j+1} of all y satisfying the equation

$$S_j(y) = -\frac{B_{j+1}}{j+1}. \tag{12}$$

Let A_{m+1} be the set of all roots of the polynomial $S_m(x)$. In particular, $\{-1, 0\} \subset A_{m+1}$. We note that if $j \geq 2$ is even, then by (12) the set A_{j+1} is the same for every $m \geq j$.

2.1. Modified Faulhaber polynomials

Consider the translated Faulhaber polynomials

$$S_m^*(x) := S_m(x) + \frac{B_{m+1}}{m+1}, \ m \in \mathbb{N}_0.$$

The polynomial $S_m^*(x)$ is divisible by $x(x+1)(2x+1)$ if $m \geq 2$ is even (Faulhaber was aware of this fact only for some selected values of m, for more information see [5, 6, 11, 12]). Clearly $S_{2m}^*(x) = S_{2m}(x)$.

Setting

$$T_{2m}(x) := \frac{S^*_{2m}(x)}{x(x+1)(2x+1)} \tag{13}$$

we get the following relation (see [11]):

$$T_{2m}\Big(x - \frac{1}{2}\Big) = T_{2m}\Big(-x - \frac{1}{2}\Big) \tag{14}$$

for every $m \in \mathbb{N}$, $m \geq 2$. Therefore the function $\mathbb{R} \ni x \mapsto T_{2m}\left(x - \frac{1}{2}\right)$ is even. It turns out that, likewise, every function $\mathbb{R} \ni x \mapsto S^*_{2m+1}\left(x - \frac{1}{2}\right)$, $m \in \mathbb{N}$, is even. These facts can be observed in Figures 4, 5 and 6 where the symmetries – both axial and polar – of the sets of roots of the appropriate polynomials are clearly visible. The above observations follow from the properties of an even real polynomial $q(x)$: if a complex number z is a root of multiplicity k of $q(x)$ then $\overline{z}$, $-z$ and $-\overline{z}$ are roots of multiplicity k of $q(x)$ as well. Moreover,

$$(m+1)S^*_m(x) = B_{m+1}(x+1),$$

where $B_m(x)$ denotes the m-th Bernoulli polynomial. This relation implies also that facts presented below, concerning the number of complex and real roots of polynomials $S^*_m(x)$, are analogical as for the case of Bernoulli polynomials $B_{m+1}(x)$ for every $m \in \mathbb{N}$.

2.2. Remarks on the roots of the polynomials $S^*_m(x)$

In this section we present several facts obtained solely by numerical computations. They concern the number of nonreal complex roots of the polynomials $S^*_n(x)$ and the number of the real roots of $S^*_n(x)$. We observed and described some rules for the growth of the number of the roots.

If $5 \leq m \leq 10$ then polynomials $S^*_m(x)$ possess only four complex roots: $c_{m,1}$, $\overline{c}_{m,1}$, $c_{m,2}$, $\overline{c}_{m,2}$, such that

$$\operatorname{Im} c_{m,1} = \operatorname{Im} c_{m,2} \quad \text{and} \quad \operatorname{Re} c_{m,2} = -1 - \operatorname{Re} c_{m,1}. \tag{15}$$

All the other roots are real. The subsequent polynomials have similar properties: for $11 \leq m \leq 15$ the polynomials $S^*_m(x)$ have two quadruples of roots of type (15), whereas for $16 \leq m \leq 20$ there are three quadruples of roots of this type and so on.

Moreover for each $m \in \mathbb{N}$ the number of nonreal complex roots of the polynomial $S^*_m(x)$ either is the same or increases by 4 in comparison with the number of nonreal complex roots of the polynomial $S^*_{m-1}(x)$. However this property cannot be treated as a rule. Starting from $S^*_0(x)$, the results of numerical computations of the number of nonreal complex roots of the successive polynomials $S^*_n(x)$ are presented below. Each row refers to 21 consecutive polynomials; furthermore we use the following notation: the number a of successive polynomials $S^*_n(x)$ $\times$ the number b of nonreal complex roots possessed by each of a consecutive polynomials. For instance the starting element 5×0 means that each of polynomials $S^*_0, \ldots, S^*_4$ has no nonreal complex roots.

The table of nonreal complex roots of $\mathbf{S_n^*(x)}$, $\mathbf{n \le 1024}$

(Polynomials are counted consecutively in rows, from left to right, according to the first factor of product $a \times b$. An element $a \times b$ of the table corresponds to a successive polynomials $S_n^*(x)$, all of which have b nonreal complex roots).

5×0,	6×4,	5×8,	5×12,	
6×16,	5×20,	5×24,	5×28,	
6×32,	5×36,	5×40,	5×44,	
6×48,	5×52,	5×56,	$\underline{\mathbf{5}} \times \underline{\mathbf{60}}$,	$\longleftarrow$ This symbol means that each
5×64,	6×68,	5×72,	5×76,	of five successive polynomials
5×80,	5×84,	6×88,	5×92,	$S_{79}^*, S_{80}^*, \ldots, S_{83}^*$ possesses
5×96,	5×100,	6×104,	5×108,	the same number of 60
5×112,	5×116,	6×120,	5×124,	nonreal complex roots.
5×128,	5×132,	5×136,	6×140,	
5×144,	5×148,	5×152,	6×156,	
$\mathbf{5} \times 160$,	$\mathbf{5} \times 164$,	$\mathbf{5} \times 168$,	$\mathbf{5} \times 172$,	$\longleftarrow$ 11th row
6×176,	5×180,	5×184,	5×188,	
6×192,	5×196,	5×200,	5×204,	
6×208,	5×212,	5×216,	5×220,	
5×224,	6×228,	5×232,	5×236,	
5×240,	6×244,	5×248,	5×252,	
5×256,	5×260,	6×264,	5×268,	
5×272,	5×276,	6×280,	5×284,	
5×288,	5×292,	5×296,	6×300,	
5×304,	5×308,	5×312,	6×316,	
5×320,	5×324,	5×328,	6×332,	
$\mathbf{5} \times 336$,	$\mathbf{5} \times 340$,	$\mathbf{5} \times 344$,	$\mathbf{5} \times 348$,	$\longleftarrow$ 22nd row
6×352,	5×356,	5×360,	5×364,	
6×368,	5×372,	5×376,	5×380,	
5×384,	6×388,	5×392,	5×396,	
5×400,	6×404,	5×408,	5×412,	
5×416,	5×420,	6×424,	5×428,	
5×432,	5×436,	6×440,	5×444,	
5×448,	5×452,	5×456,	6×460,	
5×464,	5×468,	5×472,	6×476,	
$\mathbf{5} \times 480$,	$\mathbf{5} \times 484$,	$\mathbf{5} \times 488$,	$\mathbf{5} \times 492$,	$\longleftarrow$ 31st row
6×496,	5×500,	5×504,	5×508,	
6×512,	5×516,	5×520,	5×524,	
6×528,	5×532,	5×536,	5×540,	
5×544,	6×548,	5×552,	5×556,	
5×560,	6×564,	5×568,	5×572,	
5×576,	5×580,	6×584,	5×588,	
5×592,	5×596,	6×600,	5×604,	
5×608,	5×612,	5×616,	6×620,	
5×624,	5×628,	5×632,	6×636,	

$\mathbf{5} \times 640$, $\mathbf{5} \times 644$, $\mathbf{5} \times 648$, $\mathbf{5} \times 652$, $\longleftarrow$ 41st row
6×656, 5×660, 5×664, 5×668,
6×672, 5×676, 5×680, 5×684,
5×688, 6×692, 5×696, 5×700,
5×704, 6×708, 5×712, 5×716,
5×720, 6×724, 5×728, 5×732,
5×736, 5×740, 6×744, 5×748,
5×752, 5×756, 6×760, 5×764,
5×768, 5×772, 5×776, 6×780.

Remark 2.1. The calculations were performed to a precision of 30000 sign digits.[4]

Rules of increase in the number of nonreal complex roots of polynomials $\mathbf{S_n^*(x)}$

1° The symbol $5 \times \ldots$ occurs three or four times in each row of the table presented above. Moreover, the symbol $6 \times \ldots$ appears in all rows except for the 11th, 22nd, 31st and 41st rows. In other words, every row of the table of nonreal complex roots of polynomials $S_n^*(x)$, $n \leq 1024$, contains 21 successive polynomials $S_n^*(x)$, except for the 11th, 22nd, 31st and 41st rows where there are 20 of them.

2° Let $\operatorname{ncr}(S_n^*(x))$ stand for the number of nonreal complex roots of polynomial $S_n^*(x)$, for every $n \in \mathbb{N}$. Then either $\operatorname{ncr}\left(S_{n+1}^*(x)\right) = \operatorname{ncr}(S_n^*(x))$ or $\operatorname{ncr}\left(S_{n+1}^*(x)\right) = 4 + \operatorname{ncr}(S_n^*(x))$ for each $n \in \mathbb{N}$, $n < 1024$.

3° (Generalisation of the first rule) Fix $k, n \in \mathbb{N}$. If $\operatorname{ncr}\left(S_{n+k}^*(x)\right) = 12 + \operatorname{ncr}(S_n^*(x))$ and $\operatorname{ncr}(S_n^*(x)) = 4 + \operatorname{ncr}\left(S_{n-1}^*(x)\right)$ and $\operatorname{ncr}\left(S_{n+k+1}^*(x)\right) = 4 + \operatorname{ncr}\left(S_{n+k}^*(x)\right)$ then either $k = 19$ or $k = 20$, whenever $n \geq 11$ and $k + n \leq 1024$.

We have also noticed some regularities concerning the number of real roots of polynomials $S_n^*(x)$. Namely, for the polynomials $S_n^*(x)$, $0 \leq n \leq 4$, the number of real roots increases with n from 1 to 5, in case of the polynomials $S_n^*(x)$, $7 \leq n \leq 11$, the number of real roots increases with n from 4 to 8, etc. To denote this we shall use an arrow i.e. the symbol $\nearrow$.

[4] The reason for using such a high accuracy is the arithmetic nature of the Bernoulli numbers. Namely, let $B_{2n} = N_{2n}/D_{2n}$ represent the $2n$-th Bernoulli number, where N_{2n} is the numerator, D_{2n} is the denominator and the numbers N_{2n}, D_{2n} are relatively prime, i.e. $GCD(N_{2n}, D_{2n}) = 1$. By the von Staudt-Clausen Theorem, the denominator D_{2n} is the product of distinct primes r_i such that $(r_i - 1)|2n$. It was proven in [14] that if $p > 7$ is a prime number and $2p - 1$ is also prime then

$$\left|\frac{N_{2p-2}}{N_{2p}}\right| > 910\pi^2 > 8981.$$

Note that for $p = 7$ we obtain

$$\left|\frac{N_{12}}{N_{14}}\right| = \frac{691}{7} > 10\pi^2.$$

It is a conjecture that the ratio $\left|\frac{N_{2p-2}}{N_{2p}}\right|$ for primes p is unbounded. Besides, it is known that for each $n \in \mathbb{N}$ the following inequality holds (see [1]):

$$\frac{2(2n)!}{(2\pi)^{2n}} < |B_{2n}| < \frac{2^{2n-1}}{2^{2n-1} - 1} \cdot \frac{2(2n)!}{(2\pi)^{2n}}.$$

Let us emphasize that there is an interesting Akiyama-Tanigawa algorithm for computing Bernoulli numbers in the analogous way like in Pascal's triangle for computing binomial coefficients – see [10].

We present these observations in the form of a table similar to the one for the nonreal complex roots.

The table of real roots of successive polynomials $\mathbf{S_n^*(x)}$, $\mathbf{n \leq 1024}$

$5 \times (1 \nearrow 5)$,	$6 \times (2 \nearrow 7)$,	$5 \times (4 \nearrow 8)$,	$5 \times (5 \nearrow 9)$,	
$5 \times (6 \nearrow 10)$,	$6 \times (7 \nearrow 12)$,	$5 \times (9 \nearrow 13)$,	$5 \times (10 \nearrow 14)$,	
$5 \times (11 \nearrow 15)$,	$6 \times (12 \nearrow 17)$,	$5 \times (14 \nearrow 18)$,	$5 \times (15 \nearrow 19)$,	
$5 \times (16 \nearrow 20)$,	$6 \times (17 \nearrow 22)$,	$5 \times (19 \nearrow 23)$,	$5 \times (20 \nearrow 24)$,	
$5 \times (21 \nearrow 25)$,	$6 \times (22 \nearrow 27)$,	$5 \times (24 \nearrow 28)$,	$5 \times (25 \nearrow 29)$,	
$5 \times (26 \nearrow 30)$,	$5 \times (27 \nearrow 31)$,	$6 \times (28 \nearrow 33)$,	$5 \times (30 \nearrow 34)$,	
$5 \times (31 \nearrow 35)$,	$5 \times (32 \nearrow 36)$,	$6 \times (33 \nearrow 38)$,	$5 \times (35 \nearrow 39)$,	
$5 \times (36 \nearrow 40)$,	$5 \times (37 \nearrow 41)$,	$6 \times (38 \nearrow 43)$,	$5 \times (40 \nearrow 44)$,	
$5 \times (41 \nearrow 45)$,	$5 \times (42 \nearrow 46)$,	$5 \times (43 \nearrow 47)$,	$6 \times (44 \nearrow 49)$,	
$5 \times (46 \nearrow 50)$,	$5 \times (47 \nearrow 51)$,	$5 \times (48 \nearrow 52)$,	$6 \times (49 \nearrow 54)$,	
$\mathbf{5} \times (51 \nearrow 55)$,	$\mathbf{5} \times (52 \nearrow 56)$,	$\mathbf{5} \times (53 \nearrow 57)$,	$\mathbf{5} \times (54 \nearrow 58)$,	$\leftarrow 11th$ row
$6 \times (55 \nearrow 60)$,	$5 \times (57 \nearrow 61)$,	$5 \times (58 \nearrow 62)$,	$5 \times (59 \nearrow 63)$,	
$6 \times (60 \nearrow 65)$,	$5 \times (62 \nearrow 66)$,	$5 \times (63 \nearrow 67)$,	$5 \times (64 \nearrow 68)$,	
$6 \times (65 \nearrow 70)$,	$5 \times (67 \nearrow 71)$,	$5 \times (68 \nearrow 72)$,	$5 \times (69 \nearrow 73)$,	
$5 \times (70 \nearrow 74)$,	$6 \times (71 \nearrow 76)$,	$5 \times (73 \nearrow 77)$,	$5 \times (74 \nearrow 78)$,	
$5 \times (75 \nearrow 79)$,	$6 \times (76 \nearrow 81)$,	$5 \times (78 \nearrow 82)$,	$5 \times (79 \nearrow 83)$,	
$5 \times (80 \nearrow 84)$,	$5 \times (81 \nearrow 85)$,	$6 \times (82 \nearrow 87)$,	$5 \times (84 \nearrow 88)$,	
$5 \times (85 \nearrow 89)$,	$5 \times (86 \nearrow 90)$,	$6 \times (87 \nearrow 92)$,	$5 \times (89 \nearrow 93)$,	
$5 \times (90 \nearrow 94)$,	$5 \times (91 \nearrow 95)$,	$5 \times (92 \nearrow 96)$,	$6 \times (93 \nearrow 98)$,	
$5 \times (95 \nearrow 99)$,	$5 \times (96 \nearrow 100)$,	$5 \times (97 \nearrow 101)$,	$6 \times (98 \nearrow 103)$,	
$5 \times (100 \nearrow 104)$,	$5 \times (101 \nearrow 105)$,	$5 \times (102 \nearrow 106)$,	$6 \times (103 \nearrow 108)$,	
$\mathbf{5} \times (105 \nearrow 109)$,	$\mathbf{5} \times (106 \nearrow 110)$,	$\mathbf{5} \times (107 \nearrow 111)$,	$\mathbf{5} \times (108 \nearrow 112)$,	$\leftarrow 22nd$ row
$6 \times (109 \nearrow 114)$,	$5 \times (111 \nearrow 115)$,	$5 \times (112 \nearrow 116)$,	$5 \times (113 \nearrow 117)$,	
$6 \times (114 \nearrow 119)$,	$5 \times (116 \nearrow 120)$,	$5 \times (117 \nearrow 121)$,	$5 \times (118 \nearrow 122)$,	
$5 \times (119 \nearrow 123)$,	$6 \times (120 \nearrow 125)$,	$5 \times (122 \nearrow 126)$,	$5 \times (123 \nearrow 127)$,	
$5 \times (124 \nearrow 128)$,	$6 \times (125 \nearrow 130)$,	$5 \times (127 \nearrow 131)$,	$5 \times (128 \nearrow 132)$,	
$5 \times (129 \nearrow 133)$,	$5 \times (130 \nearrow 134)$,	$6 \times (131 \nearrow 136)$,	$5 \times (133 \nearrow 137)$,	
$5 \times (134 \nearrow 138)$,	$5 \times (135 \nearrow 139)$,	$6 \times (136 \nearrow 141)$,	$5 \times (138 \nearrow 142)$,	
$5 \times (139 \nearrow 143)$,	$5 \times (140 \nearrow 144)$,	$5 \times (141 \nearrow 145)$,	$6 \times (142 \nearrow 147)$,	
$5 \times (144 \nearrow 148)$,	$5 \times (145 \nearrow 149)$,	$5 \times (146 \nearrow 150)$,	$6 \times (147 \nearrow 152)$,	
$\mathbf{5} \times (149 \nearrow 153)$,	$\mathbf{5} \times (150 \nearrow 154)$,	$\mathbf{5} \times (151 \nearrow 155)$,	$\mathbf{5} \times (152 \nearrow 156)$,	$\leftarrow 31st$ row
$6 \times (153 \nearrow 158)$,	$5 \times (155 \nearrow 159)$	$5 \times (156 \nearrow 160)$,	$5 \times (157 \nearrow 161)$,	
$6 \times (158 \nearrow 163)$,	$5 \times (160 \nearrow 164)$	$5 \times (161 \nearrow 165)$,	$5 \times (162 \nearrow 166)$,	
$6 \times (163 \nearrow 168)$,	$5 \times (165 \nearrow 169)$	$5 \times (166 \nearrow 170)$,	$5 \times (167 \nearrow 171)$,	
$5 \times (168 \nearrow 172)$,	$6 \times (169 \nearrow 174)$	$5 \times (171 \nearrow 175)$,	$5 \times (172 \nearrow 176)$,	
$5 \times (173 \nearrow 177)$,	$6 \times (174 \nearrow 179)$	$5 \times (176 \nearrow 180)$,	$5 \times (177 \nearrow 181)$,	
$5 \times (178 \nearrow 182)$,	$5 \times (179 \nearrow 183)$	$6 \times (180 \nearrow 185)$,	$5 \times (182 \nearrow 186)$,	
$5 \times (183 \nearrow 187)$,	$5 \times (184 \nearrow 188)$	$6 \times (185 \nearrow 190)$,	$5 \times (187 \nearrow 191)$,	
$5 \times (188 \nearrow 192)$,	$5 \times (189 \nearrow 193)$	$5 \times (190 \nearrow 194)$,	$6 \times (191 \nearrow 196)$,	
$5 \times (193 \nearrow 197)$,	$5 \times (194 \nearrow 198)$	$5 \times (195 \nearrow 199)$,	$6 \times (196 \nearrow 201)$,	
$\mathbf{5} \times (198 \nearrow 202)$,	$\mathbf{5} \times (199 \nearrow 203)$	$\mathbf{5} \times (200 \nearrow 204)$,	$\mathbf{5} \times (201 \nearrow 205)$,	$\leftarrow 41st$ row

$$
\begin{aligned}
&6\times(202\nearrow 207),\ 5\times(204\nearrow 208)\ 5\times(205\nearrow 209),\ 5\times(206\nearrow 210),\\
&6\times(207\nearrow 212),\ 5\times(209\nearrow 213)\ 5\times(210\nearrow 214),\ 5\times(211\nearrow 215),\\
&5\times(212\nearrow 216),\ 6\times(213\nearrow 218)\ 5\times(215\nearrow 219),\ 5\times(216\nearrow 220),\\
&5\times(217\nearrow 221),\ 6\times(218\nearrow 223)\ 5\times(220\nearrow 224),\ 5\times(221\nearrow 225),\\
&5\times(222\nearrow 226),\ 6\times(223\nearrow 228)\ 5\times(225\nearrow 229),\ 5\times(226\nearrow 230),\\
&5\times(227\nearrow 231),\ 5\times(228\nearrow 232)\ 6\times(229\nearrow 234),\ 5\times(231\nearrow 235),\\
&5\times(232\nearrow 236),\ 5\times(233\nearrow 237)\ 6\times(234\nearrow 239),\ 5\times(236\nearrow 240),\\
&5\times(237\nearrow 241),\ 5\times(238\nearrow 242)\ 5\times(239\nearrow 243),\ 6\times(240\nearrow 245).
\end{aligned}
$$

In the following table we give examples of polynomials $S^*_m(x)$ and their roots for some selected values of $m\in\mathbb{N}_0$.

m	$S^*_m(x)$	real roots (all combinations of signs $\pm$ should be taken into account)
0	$\frac{1}{2}(2x+1)$	$-\frac{1}{2}$
1	$\frac{1}{12}(6x^2+6x+1)$	$\frac{1}{6}(-3\pm\sqrt{3})$
2	$\frac{1}{6}x(x+1)(2x+1)$	$-1, -\frac{1}{2}, 0$
3	$\frac{1}{120}(30x^4+60x^3+30x^2-1)$	$\frac{1}{30}\left(\pm\sqrt{15\left(15\pm 2\sqrt{30}\right)}-15\right)$
4	$\frac{1}{30}x(x+1)(2x+1)(3x^2+3x-1)$	$-1, -\frac{1}{2}, 0, \frac{1}{6}\left(-3\pm\sqrt{21}\right)$
10	$\frac{1}{66}x(x+1)(2x+1)(x^2+x-1)\times$ $\times(3x^6+9x^5+2x^4-11x^3+3x^2+10x-5)$	$-1, -\frac{1}{2}, 0, \frac{1}{2}(-1\pm\sqrt{5})$, $-1.51868, 0.51868$
5	$\frac{1}{252}\big(42x^6+126x^5+$ $+105x^4-21x^2+1\big)$	$-0.7524, -0.2475$
7	$\frac{1}{240}\big(30x^8+120x^7+140x^6+$ $-70x^4+20x^2-1\big)$	$-1.2472, -0.7506$ $-0.2493, 0.2472$
9	$\frac{1}{660}\big(66x^{10}+330x^9+495x^8+$ $-462x^6+330x^4-99x^2+5\big)$	$-1.5739, -1.2499, -0.75015$ $-0.2498, 0.2499, 0.5739$

Rules of increase in the number of real roots of polynomials $\mathbf{S^*_n(x)}$

1° As in the case of nonreal complex roots, $5\times\ldots$ appears three or four times in each row of the table of real roots of $S^*_n(x)$ and the symbol $6\times\ldots$ occurs in all rows except the 11th, 22nd, 31st and 41st rows.

2° Let $\operatorname{rr}(S^*_n(x))$ stand for the number of real roots of polynomial $S^*_n(x)$. Then either $\operatorname{rr}\left(S^*_{n+1}(x)\right)=1+\operatorname{rr}(S^*_n(x))$ or $\operatorname{rr}\left(S^*_{n+1}(x)\right)=\operatorname{rr}(S^*_n(x))-3$ for each $n\in\mathbb{N}$, $n<1024$. Furthermore, if $\operatorname{rr}\left(S^*_{n+k}(x)\right)=k+\operatorname{rr}(S^*_n(x))$ and simultaneously $\operatorname{rr}\left(S^*_{n+k+1}(x)\right)<\operatorname{rr}\left(S^*_{n+k}(x)\right)$ and $\operatorname{rr}\left(S^*_{n-1}(x)\right)>\operatorname{rr}(S^*_n(x))$ then either $k=4$ or $k=5$ for every $k,n\in\mathbb{N}$, $k+n\le 1024$.

3° In each row of the table of real roots of polynomials $S^*_n(x)$ there are either 20 or 21 successive polynomials $S^*_n(x)$.

Now, for each $m \in \mathbb{N}$, set

$$T_{2m+1}(x) := \frac{S_{2m+1}(x)}{x^2(x+1)^2}.$$

Note that the functions $T_{2m}(x)$ have been already defined in (13).

Now all the coefficients of $T_n(x)$ are rational, so one can express them as irreducible fractions, multiply $T_n(x)$ by their least common denominator and then divide, if necessary, by the greatest common factor of the numerators of the fractions. Then we obtain the new polynomial denoted henceforth by $T_n^*(x)$, all coefficients of which are integers with the greatest common divisor equal to 1. Furthermore, $T_n^*(x)$ can be treated as a polynomial in variable (x^2+x-a) for every $a \in \mathbb{C}$ and $n \in \mathbb{N}$, $n \geq 4$ (see [11]). The sequence $\{a_n\}$, where $a_n := A251926(n) :=$ the remainder in the division of $T_n^*(x)$ by x^2+x-1 for every $n \geq 4$ ($A251926$ means the number of the respective sequence in Sloane's OEIS) is of a special interest since a_n is equal also to the remainder in the division of $Q_n(x)$ by x^2+x-1 for every $n \geq 4$, where

$$Q_n(x) := \begin{cases} S_n(x), & \text{if } n \in 2\mathbb{N}-1, \\ \frac{S_n(x)}{2x+1}, & \text{if } n \in 2\mathbb{N}. \end{cases}$$

The first 25 elements of this sequence are: 2, 1, 1, 1, 1, 0, 0, 1, 37, −60, −5, 37, 174, −955, −10545, 38610, 176297, −322740, −205420, 4512655, 56820585, −104019264, −25907081, 94854194, 1141847218.

Below we present the triangle of coefficients of polynomials $T_n^*(x)$, for $4 \leq n \leq 17$, treated as the polynomials in the variable x^2+x-1:

$$\begin{array}{rrrrrrrr}
2, & 3, & & & & & & \\
1, & 2, & & & & & & \\
1, & 3, & 3, & & & & & \\
1, & 2, & 3, & & & & & \\
1, & 4, & 5, & 5, & & & & \\
0, & 2, & 1, & 2, & & & & \\
0, & 1, & 5, & 2, & 3, & & & \\
1, & -2, & 5, & 0, & 2, & & & \\
37, & 83, & -155, & 385, & 0, & 105, & & \\
-60, & 194, & -208, & 174, & -25, & 30, & & \\
-5, & -8, & 38, & -34, & 24, & -3, & 3, & \\
37, & -114, & 139, & -84, & 37, & -6, & 3, & \\
174, & 291, & -1250, & 1300, & -655, & 245, & -35, & 15, \\
-955, & 2954, & -3558, & 2244, & -855, & 240, & -35, & 10.
\end{array}$$

For example the sequence $1, -2, 5, 0, 2$ (8th row of the triangle) corresponds to the decomposition

$$T_{11}^*(x) = 1 - 2(x^2+x-1) + 5(x^2+x-1)^2 + 2(x^2+x-1)^4.$$

We also observe the specific relations

$$T_4^*(x) - T_5^*(x) = x^2 + x \quad \Rightarrow \quad a_2 - a_3 = 1,$$

$$T_6^*(x) - T_7^*(x) = x^2 + x - 1 \quad \Rightarrow \quad a_4 = a_5,$$
$$T_9^*(x) = (x^2 + x - 1)(2x^4 + 4x^3 - x^2 - 3x + 3) \quad \Rightarrow \quad a_7 = 0.$$

The remainder R_n in the division of $T_n^*(x)$ by $x^2 + x$ has been discussed by D.E. Knuth in [11]. In Section 7 of [11] he proved that R_n is equal to the numerator of $\binom{2n}{2} B_{2(n-1)}$ for every $n \in \mathbb{N}$.

2.3. Figures

In the following figures there are presented the distributions of nonreal complex roots which we will refer to as "complex roots" for short of selected polynomials $S_m^*(x)$ and $S_m^*\left(x - \frac{1}{2}\right)$.

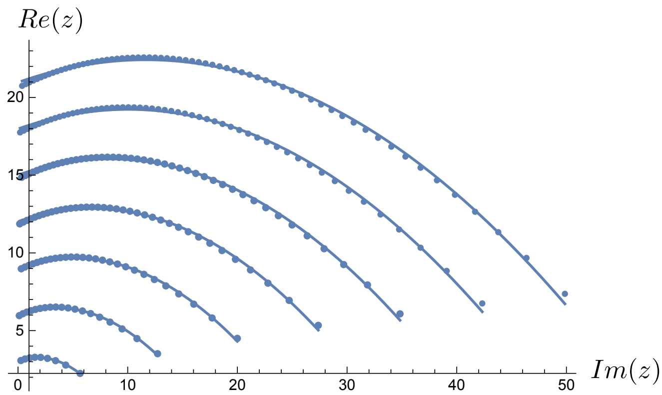

Fig. 1. Parabolas approximating the locations of the complex roots of $S_m^*(x)$ from positive quadrant (i.e. $Re(z) > 0$ and $Im(z) > 0$) for selected values of m

In Fig. 1 the respective parabolas have the following equations (starting from the bottom):

$m = 50$: $y = 3.0426 + 0.25066x - 0.069745x^2$;
$m = 100$: $y = 6.0301 + 0.25109x - 0.035904x^2$;
$m = 150$: $y = 9.0404 + 0.24541x - 0.024104x^2$;
$m = 200$: $y = 12.013 + 0.24901x - 0.018409x^2$;
$m = 250$: $y = 15.009 + 0.24801x - 0.014823x^2$;
$m = 300$: $y = 17.976 + 0.25073x - 0.012489x^2$;
$m = 350$: $y = 20.968 + 0.25036x - 0.010751x^2$.

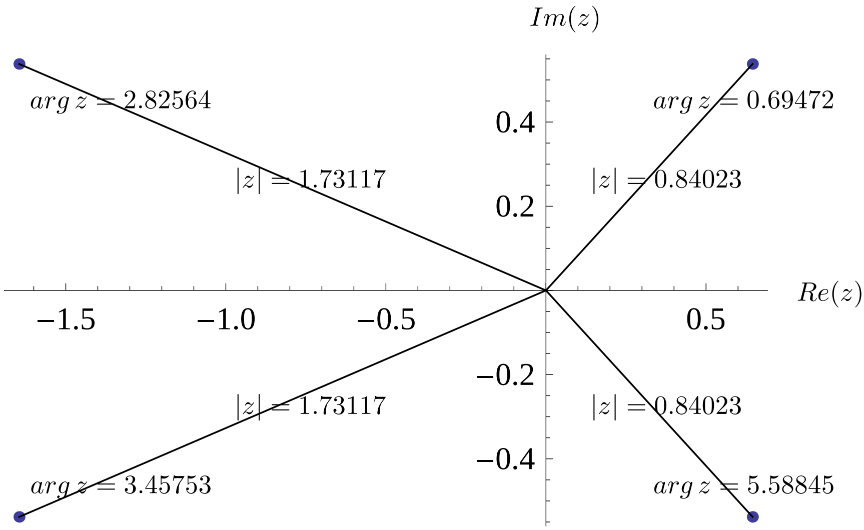

Fig. 2. Distribution of the complex roots of $S^*_{10}(x)$

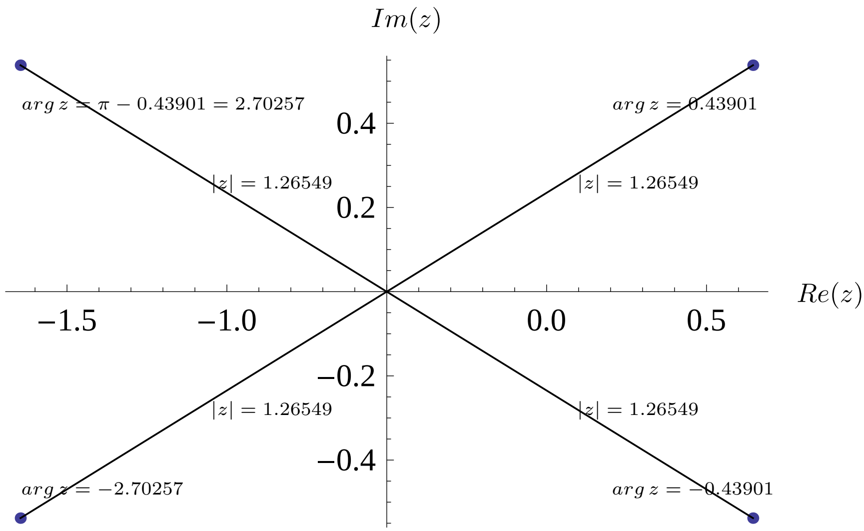

Fig. 3. Distribution of the complex roots of $S^*_{10}(x - \frac{1}{2}) = \frac{1}{11} B_{11}(x + \frac{1}{2})$

In Fig. 4–6 the external collection of numbers represents the moduli of the respective roots, whereas the internal collection represents their arguments.

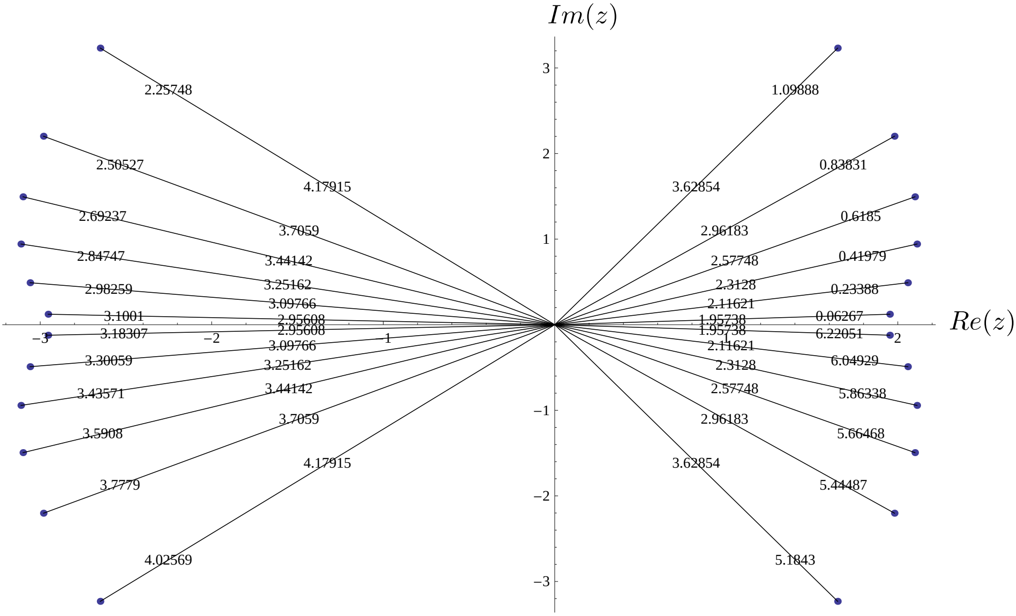

Fig. 4. The distribution of the complex roots of $S_{32}^*(x)$

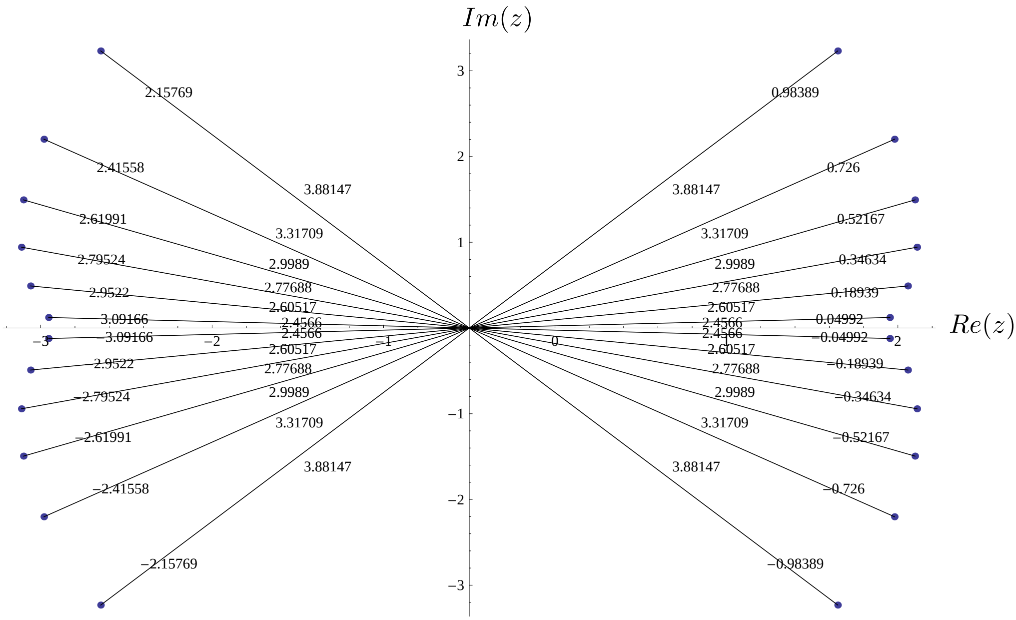

Fig. 5. Distribution of the complex roots of $S_{32}^*(x-\frac{1}{2}) = \frac{1}{33}B_{33}(x+\frac{1}{2})$

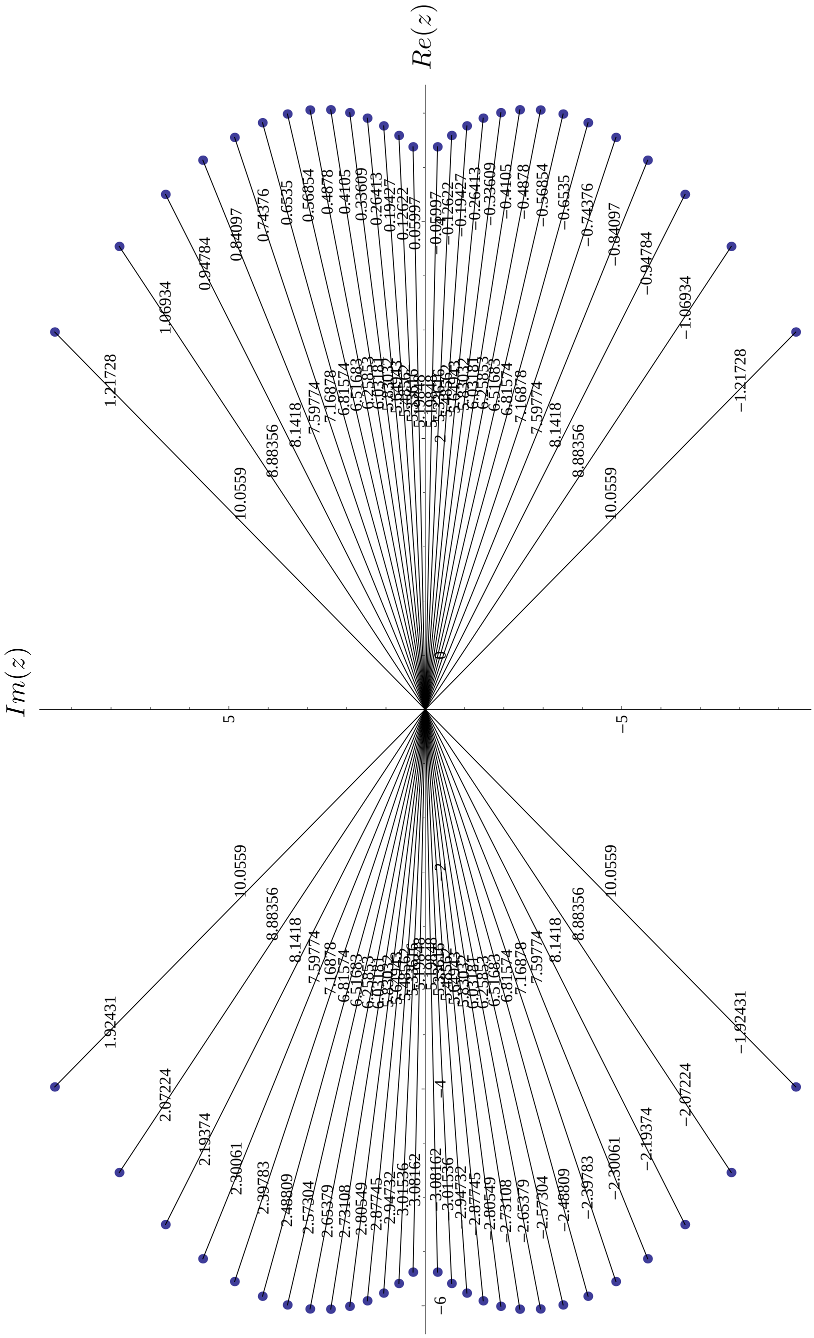

Fig. 6. Distribution of the complex roots of $S^*_{77}(x-\frac{1}{2}) = \frac{1}{78}B_{78}(x+\frac{1}{2})$

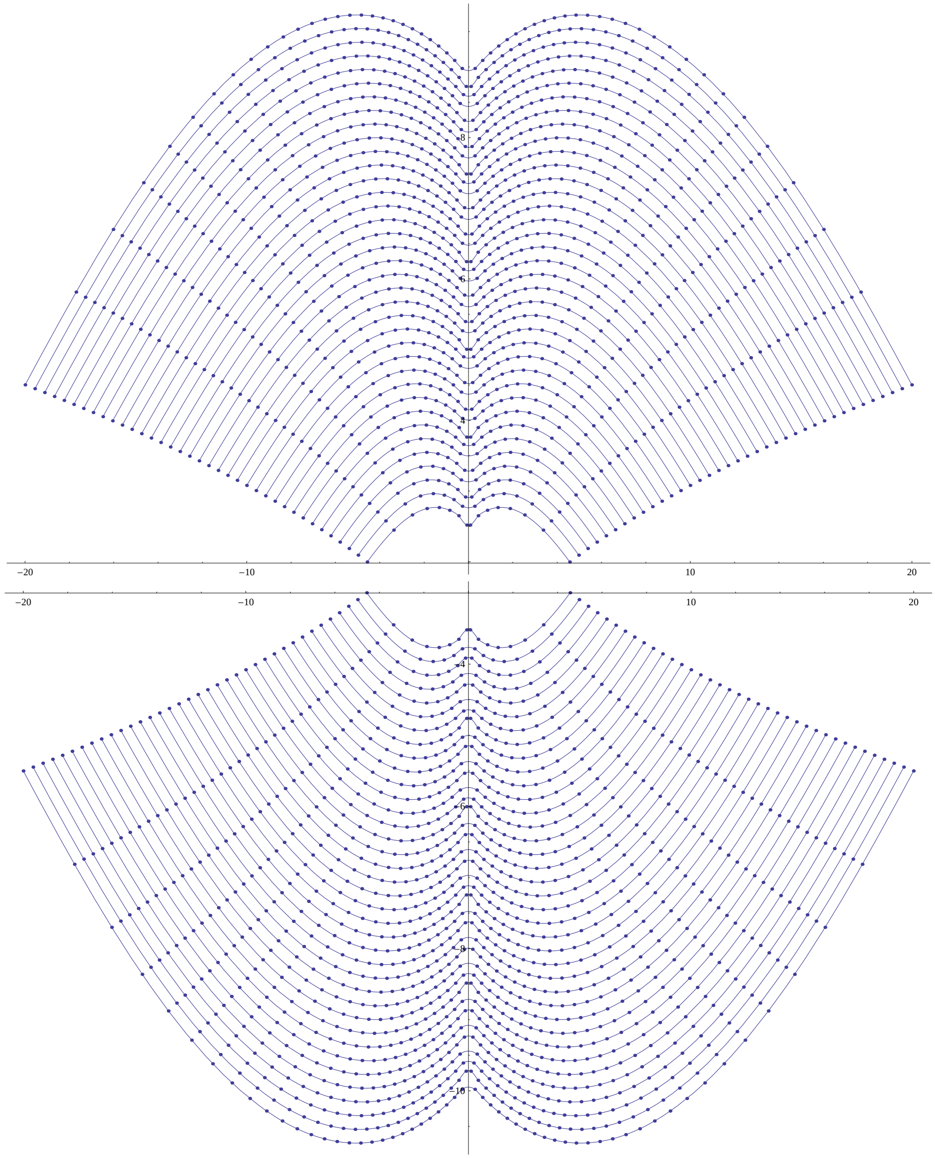

Fig. 7. Distribution of the complex roots of polynomials $S^*_{42+3m}(x)$, $0 \le m \le 36$ (for the single polynomial one should take into account the respective pairs of lines symmetric with respect to the horizontal axis). Note that the imaginary axis is horizontal, and the real axis is vertical

Bibliography

1. Abramowitz M., Stegun I.A.: *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. U.S. Government Printing Office, Washington 1964.
2. Bazsó A., Pintér Á., Srivastava H.M.: *A refinement of Faulhaber's theorem concerning sums of powers of natural numbers*. Applied Math. Letters **25** (2012), 486–489.
3. Chen W.Y.C., Fu A.M., Zhang J.F.: *Faulhaber's theorem on power sums*. Discrete Math. **309** (2009), 2974–2981.
4. Dilcher K.: *Bernoulli and Euler Polynomials*. In: NIST Handbook of Mathematical Functions, F.W.I. Olver, D.W. Lozier, R.F. Boisvert, C.W. Clark (eds.). Cambridge Univ. Press, Cambridge 2010.
5. Edwards A.W.F.: *A Quick Route to Sums of Powers*. Amer. Math. Monthly **93**, no. 6 (1986), 451–455.
6. Faulhaber J.: *Academia Algebrae, Darinnen die miraculosische Inventiones zu den höchsten Cossen Weiters continuirt und profitiert warden*. Johann Ulrich Schönigs, Augspurg 1631.
7. Graham R.L., Knuth D.E., Patashnik O.: *Concrete Mathematics*. Addison Wesley Publishing Company, Reading 1994.
8. Hetmaniok E., Lorenc P., Pleszczyński M., Wituła R.: *Iterated integrals of polynomials*. Applied Math. Comp. **249** (2014) 389–398.
9. Ivić A.: *The Riemann Zeta-Function. Theory and Applications*. Dover Publications, New York 2003.
10. Kaneko M.: *The Akiyama-Tanigawa algorithm for Bernoulli numbers*. J. Integer Seq. **3** (2000), Article 00.2.9.
11. Knuth D.E.: *Johann Faulhaber and the sums of powers*. Math. Comput. **203** (1993), 277–294.
12. Krishnapriyan H.K.: *Polynomials and Faulhaber's result on sums of power of integers*. College Math. J. **26**, no. 2 (1995), 118–123.
13. Lehmer D.H.: *On the maxima and minima of Bernoulli polynomials*. Amer. Math. Monthly **47**, no. 8 (1940), 533–538.
14. Powell B.J., Frame J.S.: *Primes, ratios and Bernoulli numbers*. Amer. Math. Monthly **90**, no. 9 (1983), 645–646.
15. Rabsztyn S., Słota D., Wituła R.: *Gamma and Beta Functions*. Wyd. Pol. Śl., Gliwice 2012 (in Polish).
16. Snyder M.A.: *A new set of polynomials, related to the Bernoulli numbers, which display a finite Stokes phenomenon*. Integral Transforms Spec. Funct. **25**, no. 2 (2014), 124–133.